Journal of Mathematical Sciences, Vol. 131, No. 2, 2005

A NEW APPROACH TO THE REPRESENTATION THEORY OF THE SYMMETRIC GROUPS. II

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The present paper is a revised Russian translation of the paper "A new approach to representation theory of symmetric groups," Selecta Math., New Series, **2***, No.* 4, 581–605 (1996). *Numerous modifications to the text were made by the first author for this publication. Bibliography:* 35 *titles.*

To the memory of D. Coxeter

Preface

This paper is a revised Russian translation of a paper by the same authors (see the reference below) and is devoted to a nontraditional approach to the representation theory of the symmetric groups (and, more generally, to the representation theory of Coxeter and local groups). The translation was prepared for the Russian edition of the book W. Fulton, *Young Tableaux. With Applications to Representation Theory and Geometry*, Cambridge Univ. Press, Cambridge, 1997, which, hopefully, will appear sooner or later. In the editor's preface to the Russian translation of the book it is explained what is the drawback of the conventional approach to the representation theory of the symmetric groups: it does not take into account important properties of these groups, namely, that they are Coxeter groups, and that they form an inductive chain, which implies that the theory must be constructed inductively. A direct consequence of these drawbacks is, in particular, that Young diagrams and tableaux appear *ad hoc*; there presence in the theory is justified only after the proof of the branching theorem.

The theory described in this paper is intended to correct these defects. The first attempt in this direction was the paper [30] by the first author, in which it was proved that if we assume that the branching graph of irreducible complex representations of the symmetric groups is distributive, then it must be the Young graph. As it turned out, this *a priori* assumption is superfluous — the distributivity follows directly from the fact that S_n is a Coxeter group if we involve remarkable generators of the Gelfand–Tsetlin subalgebra of the group algebra $\mathbb{C}[S_n]$, namely, the Young[1]–Jucys[2]–Murphy generators (see [19, 30]). But all numerous later expositions, including the very good book by Fulton, followed the classical version of the theory, which goes back to Frobenius, Schur, and Young; although some nice simplifications were made, such as von Neumann's lemma, Weyl's lemma, the notion of tabloids, etc., but the general scheme of the construction of the theory remained the same.[3] The reader can find references to the books on the representation theory of the symmetric groups in the monograph by James and Kerber [18], in the book by James [17], which was translated into Russian, and in earlier textbooks.

The key point of our approach, which explains the appearance of Young tableaux as well as the general idea of our method, is that the points of the spectrum of the Gelfand–Tsetlin algebra with respect to the Young–Jucys–Murphy generators are so-called content vectors, i.e., integer vectors in $\mathbb{R}^n$ that satisfy certain simple conditions, which follow from the Coxeter relations, and the coordinates of these integer vectors are the so-called contents of the boxes of Young tableaux (see Sec. 6); since the content vector uniquely determines a Young tableau, it follows that *the points of the spectrum are precisely Young tableaux.* The corresponding eigenvectors determine a basis in each representation, and the set of vectors corresponding to tableaux with a given diagram form a basis of the irreducible representation of S_n (the Young–Gelfand–Tsetlin basis). *Thus the correspondence "diagrams" ↔ "irreducible representations" obtains a natural (one might say, spectral) explanation.*

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[1]The so-called Young's orthogonal and seminormal form for the action of the Coxeter transpositions in irreducible representations were defined in Young's last papers; apparently, he regarded them only as an illustration; these forms play an essential role in our theory (see Secs. 3, 7). Some time ago A. Lascoux observed that these generators were mentioned explicitly in Young's paper, and recently R. Stanley gave a precise reference. But apparently Young himself underestimated their importance.

[2]A.-A. A. Jucys (1936–1998) is a Lithuanian mathematician. The paper [19], where he introduced these generators, remained unnoticed for a long time; an English mathematician G. Murphy rediscovered them and then found Jucys' paper.

[3]Our approach to the representation theory of the symmetric groups was recently used in [35].

Translated from *Zapiski Nauchnykh Seminarov POMI*, Vol. 307, 2004, pp. 57–98. Original article submitted March 2, 2004.

1072-3374/05/1312-5471

Our approach not only helps to improve the exposition of classical results, it also allows us to consider representations of more general groups and algebras, for example, “local groups and algebras” in the sense of [30], provided that the group is finite or the algebra is finite-dimensional. An attempt to apply this method to other groups and, in particular, to the Coxeter groups of series B–C–D, is contained in [12] and [28].

Recently died a distinguished and original mathematician Donald Coxeter (1907–2003), to whom modern mathematics owes important and deep ideas and very beautiful geometric and group constructions. This revised version of the paper is dedicated to the memory of D. Coxeter.

A. Vershik

0. Introduction

The aim of this paper is to present a new, simple and direct, approach to the representation theory of the permutation group S_n.

Basically, there are two ways to construct irreducible complex representations of S_n. The first one is essentially based on the representation theory of the full linear group $GL(N)$ and the duality between S_n and $GL(N)$ in the space

$$\underbrace{\mathbb{C}^N \otimes \mathbb{C}^N \otimes \cdots \otimes \mathbb{C}^N}_{n \text{ times}},$$

which is called the Schur–Weyl duality (see [1]). The Schur functions, which are characters of $GL(N)$, play the key role in this approach. A description of the characters of S_n that is based on the Schur functions and is close to the original construction by Frobenius can be found, for example, in [23].

The other way, usually attributed to Young with later contributions by von Neumann and Weyl, is based on the combinatorics of tableaux. In this approach, an irreducible representation (sometimes called a Specht module) arises as the unique common component of two simple representations induced from one-dimensional representations (the identity representation and the sign representation) of the same Young subgroup. It is this irreducible component that one associates with the partition (diagram) corresponding to the Young subgroup. Since the decomposition of induced representations into irreducible ones is rather complicated and nonconstructive, the correspondence “diagrams” $\leftrightarrow$ “irreducible representations” also looks rather unnatural. This approach is traditional, and one can find it in almost all textbooks and monographs on the subject, for example, in one of the last books [18]. Under this approach, considerable efforts are required to obtain any explicit formula for characters of S_n.

Both these ways are important as well as indirect; they rest upon deep and nontrivial auxiliary constructions. There is a natural question: whether one can arrive at the main combinatorial objects of the theory (diagrams, tableaux, etc.) in a more direct and natural fashion?

We believe that the representation theory of the symmetric groups must satisfy the following three conditions:

(1) The symmetric groups form a natural chain (S_{n-1} is embedded into S_n), and the representation theory of these groups should be constructed inductively with respect to these embeddings, that is, the representation theory of S_n should rely on the representation theory of S_{n-1}, $n = 1, 2, \ldots$.

(2) The combinatorics of Young diagrams and Young tableaux, which reflects the branching rule for the restriction

$$S_n \downarrow S_{n-1},$$

should be introduced as a natural auxiliary element of the construction rather than *ad hoc*; it should be deduced from the intrinsic structure of the symmetric groups. Only in this case the branching rule (which is one of the main theorems of the theory) appears naturally and not as a final corollary of the whole theory.

(3) The symmetric groups are Coxeter groups, and the methods of their representation theory should apply to all classical series of Coxeter groups.

In this paper, we suggest a new approach, which satisfies the above principles and makes the whole theory more natural and simple. The following notions are very important for our approach:

(1) Gelfand–Tsetlin algebra and Gelfand–Tsetlin basis (GZ-algebra and GZ-basis);
(2) Young–Jucys–Murphy (YJM-) elements;
(3) algebras with a local system of generators (ALSG) as a general context for the theory.

The Gelfand–Tsetlin basis was defined by I. M. Gelfand and M. L. Tsetlin in the fifties [5, 6] for the unitary and orthogonal groups. The general notion of GZ-algebra for inductive limits of algebras can be introduced in the same way for an arbitrary inductive limit of semisimple algebras (this was done, for example, in [3]). For the general definition of Gelfand–Tsetlin algebras and Young–Jucys–Murphy generators, see also [34].

The notion of algebras or groups with a local system of generators and local relations (in short, local algebras or groups) generalizes Coxeter groups, braid groups, Hecke algebras, locally free algebras, etc. (see [30, 31]). This notion allows one to define an inductive process of constructing representations, which we apply here to the symmetric groups.

The special generators of the GZ-algebra of the symmetric group S_n were essentially introduced in papers by A. Young and then rediscovered independently by A.-A. A. Jucys [19] and G. E. Murphy [24]. These YJM-generators are as follows:

$$X_i = (1\,i) + (2\,i) + \cdots + (i-1\,i), \quad i = 1, 2, \ldots, n;$$
$$X_0 = 0, \quad X_1 = (1, 2), \quad \ldots$$

There exist an invariant way to define them (see below), which applies to a very general class of ALSG, in particular, to all Coxeter groups. It is very important that these generators do not lie in the centers of the corresponding group algebras, but nevertheless generate the GZ-algebra, which contains all these centers.

The complexity of the symmetric group (compared, for example, to the full linear group) lies in the fact that the Coxeter relations

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

for the generators s_i of S_k are not commutation relations. Moreover, there is no sufficiently large commutative subgroup of S_k that could play the role of a Cartan subgroup. However, our approach in some way resembles Cartan's highest weight theory, with the role of a Cartan subgroup played by the commutative GZ-subalgebra in $\mathbb{C}[S_n]$. The Young–Jucys–Murphy generators of this subalgebra diagonalize simultaneously in any representation of S_n, and the whole representation theory of S_n is encoded in their spectrum. The problem is, therefore, to describe this spectrum, that is, to understand what eigenvalues of the YJM-elements can appear and which of them appear in a given irreducible representation.

This problem is similar to the description of the dominant weights of a reductive group. We solve it using induction on n and elementary analysis of the commutation relation

$$s_i X_i + 1 = X_{i+1} s_i\,, \quad i = 1, 2, \ldots, n-1, \tag{0.1}$$

between the YJM-elements and the Coxeter generators s_i. In a sense, the algebra $H(2)$ (the degenerate affine Hecke algebra of order 2) generated by s_i and two commuting elements X_i and X_{i+1} subject to (0.1) plays the same role in our paper as the group $\mathfrak{gl}(2)$ plays in the representation theory of reductive groups.

Our exposition is organized as follows. We define the branching scheme of irreducible representations of the symmetric groups S_n and prove that it is a graph (rather than a multigraph), i.e., the multiplicities of irreducible representations of S_{n-1} in the restrictions of irreducible representations of S_n to S_{n-1} are simple. Then we study a maximal commutative subalgebra of the group algebra — the Gelfand–Tsetlin algebra, or the GZ-algebra, whose diagonalization in each irreducible representation determines a linear basis of this representation, and show that the spectrum of this algebra is the set of integer vectors in $\mathbb{R}^n$ determined by simple conditions described in Sec. 5 (so-called content vectors). A vector satisfying these conditions is in turn just the vector consisting of the "contents" of the boxes of a Young tableau (such a vector uniquely determines the tableau), and thus we arrive at the main conclusion that the bases of all irreducible complex representations of S_n are indexed by Young tableaux. There is an equivalence relation on content vectors: two vectors are equivalent if they belong to the same irreducible representation. We prove that this equivalence of the corresponding tableaux means that they have the same Young diagram, and this completes the proof of the main theorem — the branching theorem: the branching graph (Bratteli diagram) of irreducible representations of the symmetric groups S_n coincides with the graph of Young diagrams (the Young graph).

Two facts allow us to realize this plan: first, we choose the so-called Young–Jucys–Murphy generators of the Gelfand–Tsetlin algebra and consider the spectrum with respect to these generators; and, second, we can explicitly describe the representations of the degenerate affine Hecke algebra $H(2)$, which plays the role of the "increment" in the inductive step from the group algebra $\mathbb{C}[S_{n-1}]$ to the group algebra $\mathbb{C}[S_n]$. This step can be realized because of the role that is played by the Coxeter generators of S_n and the Coxeter relations between them: they directly give conditions on elements of the spectrum of the GZ-algebra (content vectors). One of the main advantages of our construction of the representation theory of the symmetric groups (and other series of Coxeter groups) is that *we obtain the branching rule simultaneously with the description of representations, and introduce Young diagrams and tableaux using only the analysis of the spectrum of the GZ-algebra.* One may say that our plan also realizes a noncommutative version of Fourier analysis on the symmetric groups, in which *the set of Young tableaux appears in a natural way as the spectrum of a dual object to S_n, and the set of diagrams gives the list of representations.*

As an application of these results, we derive the classical Young formulas for the action of the Coxeter generators s_i of S_n and a new proof of the Murnaghan–Nakayama rule for the characters of S_n. The final step in the proof of the Young formulas is the same as in [24]; in fact, the derivation of the Young formulas was Murphy's motivation for introducing the elements X_i. The novelty of our approach compared to [24] is that we do not assume any knowledge of the representation theory of S_n and, on the contrary, construct the theory starting from simple commutation relations.[4]

The first attempt to develop a new approach to the representation theory of the symmetric groups was made in the papers [30, 31], where the notion of algebras with a local system of generators (ALSG) was introduced. The branching rule and Young's orthogonal form were deduced in [30] from the Coxeter relation for the generators of S_n and the assumption that the branching graph (see below) of S_n is the Hasse diagram of a *distributive lattice*. The approach presented in this paper does not require any additional assumptions.

Our scheme can be applied to some other ALSG, and first of all to the Coxeter groups of B–C–D series and to wreath products of the symmetric groups with some finite groups. All these generalizations will be considered elsewhere.

We do not attempt to give a complete bibliography on the subject. Proper analogs of the Young–Jucys–Murphy elements for the infinite symmetric group S_∞ proved to be an extremely powerful tool in infinite-dimensional representation theory; see [8, 9, 10, 11]. For the representation theory of the infinite symmetric group, see also [20, 32, 3, 21]. In the series of papers [30, 31, 32], the first author develops a new approach to the representation theory of S_n in connection with asymptotic problems.

There are numerous other applications of the YJM-elements to classical representation theory (see, for example, [15]; we learned about this important preprint after our paper was completed). The Young–Jucys–Murphy elements arise naturally in connection with higher Capelli identities (see [27]). In [13, 16], these elements were considered in the context of the theory of degenerate affine Hecke algebras. Young–Jucys–Murphy elements for Coxeter groups were defined in [26, 28]; among earlier papers, we mention [7].

In what follows, the reader is supposed to be familiar only with elementary facts from the abstract representation theory of finite groups. We will not use any facts from the representation theory of the symmetric groups.

A short announcement of our results was made in [4].

1. Gelfand-Tsetlin algebra and Gelfand–Tsetlin basis

Consider an inductive chain of finite groups

$$\{1\} = G(0) \subset G(1) \subset G(2) \subset \dots . \tag{1.1}$$

By $G(n)^\wedge$ denote the set of equivalence classes of irreducible complex representations of the group $G(n)$. By definition, the *branching graph* (more precisely, the *branching multigraph*), also called the *Bratteli diagram*, of this chain is the following directed graph. Its vertices are the elements of the set (disjoint union)

$$\bigcup_{n \geq 0} G(n)^\wedge .$$

[4] From the viewpoint of the classical representation theory of S_n, it may seem that using the whole inductive family, $S_1 \subset \cdots \subset S_{n-1} \subset S_n$ to construct the representation theory of the unique group S_n is somewhat arbitrary (there are many such families, although they are isomorphic). But it is this "noninvariance" that allows us to relate the theory to Young diagrams and tableaux; without it there is no branching theorem, no GZ-bases, no RSK correspondence, etc. Moreover, without fixing an inductive family, the correspondence "irreducible representations" $\leftrightarrow$ "Young diagrams" loses its precise sense and remains only an arbitrary act of constructing the Specht modules. Of course, other inductive families (for example, $S_2 \subset S_4 \subset \dots$ with periodic embeddings) lead to other branching theorems and other bases.

Denote by V^λ the $G(n)$-module corresponding to a representation $\lambda \in G(n)^\wedge$. Two vertices $\mu \in G(n-1)^\wedge$ and $\lambda \in G(n)^\wedge$ are joined by k directed edges (from μ to λ) if

$$k = \dim \mathrm{Hom}_{G(n-1)}(V^\mu, V^\lambda)\,,$$

that is, if k is the multiplicity of μ in the restriction of λ to the group $G(n-1)$. We call the set $G(n)^\wedge$ the nth *level* of the branching graph. We write

$$\mu \nearrow \lambda$$

if μ and λ are connected by an edge in the branching graph; and

$$\mu \subset \lambda\,,$$

where $\mu \in G(k)^\wedge$, $\lambda \in G(n)^\wedge$, and $k \le n$, if the multiplicity of μ in the restriction of λ to $G(k)$ is nonzero. In other words, $\mu \subset \lambda$ if there is a path from μ to λ in the branching graph. Denote by $\varnothing$ the unique element of $G(0)^\wedge$. The same definition of the branching graph applies to any chain

$$M(0) \subset M(1) \subset M(2) \subset \ldots$$

of finite-dimensional semisimple algebras (see [3] and references therein). If the multiplicities of all restrictions are equal 0 or 1, then this diagram is a graph (and not multigraph); in this case one says that the *multiplicities are simple* or the *branching is simple*. It is well known, and we will prove this in the next section, that this is the case for the symmetric groups $G(n) = S_n$ (see also, e.g., [18, 17]). If the branching is simple, the decomposition

$$V^\lambda = \bigoplus_{\mu \in G(n-1)^\wedge,\, \mu \nearrow \lambda} V^\mu$$

into the sum of irreducible $G(n-1)$-modules is canonical. By induction, we obtain a canonical decomposition of the module V^λ into irreducible $G(0)$-modules (i.e., one-dimensional subspaces)

$$V^\lambda = \bigoplus_T V_T$$

indexed by all possible chains

$$T = \lambda_0 \nearrow \lambda_1 \nearrow \ldots \nearrow \lambda_n, \tag{1.2}$$

where $\lambda_i \in G(i)^\wedge$ and $\lambda_n = \lambda$. Such chains are *increasing paths* from $\varnothing$ to λ in the branching graph (or multigraph).

Choosing a unit (with respect to the $G(n)$-invariant inner product $(\cdot\,,\cdot)$ in V^λ) vector v_T in each one-dimensional space V_T, we obtain a basis $\{v_T\}$ in the module V^λ, which is called the *Gelfand–Tsetlin basis (GZ–basis)*. In [5, 6], such a basis was defined for representations of $SO(n)$ and $U(n)$; we use the same term in the general situation (see [3]). By the definition of v_T,

$$\mathbb{C}[G(i)] \cdot v_T, \quad i = 1, 2, \ldots, n, \tag{1.3}$$

is the irreducible $G(i)$-module V^{λ_i}. It is also clear that v_T is the unique (up to a scalar factor) vector with this property.

By $Z(n)$ denote the center of $\mathbb{C}[G(n)]$. Let $GZ(n) \subset \mathbb{C}[G(n)]$ be the algebra generated by the subalgebras

$$Z(1), Z(2), \ldots, Z(n)$$

of $\mathbb{C}[G(n)]$. It is readily seen that the algebra $GZ(n)$ is commutative. It is called the *Gelfand–Tsetlin subalgebra (GZ-algebra)* of the inductive family of (group) algebras. Recall the following fundamental isomorphism:

$$\mathbb{C}[G(n)] = \bigoplus_{\lambda \in G(n)^\wedge} \mathrm{End}\,(V^\lambda) \tag{1.4}$$

(the sum is over all equivalence classes of irreducible complex representations).

Proposition 1.1. *The algebra $GZ(n)$ is the algebra of all operators diagonal in the Gelfand–Tsetlin basis. In particular, it is a maximal commutative subalgebra of $\mathbb{C}[G(n)]$.*

Proof. Denote by $P_T \in GZ(n)$ the product

$$P_{\lambda_1} P_{\lambda_2} \dots P_{\lambda}, \quad P_{\lambda_i} \in Z(i),$$

of the central idempotents corresponding to the representations $\lambda_1, \lambda_2, \dots, \lambda$, respectively. Clearly, P_T is a projection onto V_T. Hence $GZ(n)$ contains the algebra of operators diagonal in the basis $\{v_T\}$, which is a maximal commutative subalgebra of $\mathbb{C}[G(n)]$. Since $GZ(n)$ is commutative, the proposition follows. $\square$

Remark 1.2. Note that by the above proposition, any vector from the Gelfand–Tsetlin basis in any irreducible representation of $G(n)$ is uniquely (up to a scalar factor) determined by the eigenvalues of the elements of $GZ(n)$ on this vector.

Remark 1.3. For an arbitrary inductive family of semisimple algebras, the GZ-subalgebra is a maximal commutative subalgebra if and only if the branching graph has no multiple edges.

The following criterion of simple branching uses the important notion of centralizer. Let M be a semisimple finite-dimensional $\mathbb{C}$-algebra, and let N be its subalgebra; the *centralizer* $Z(M, N)$ of this pair is the subalgebra of all elements of M that commute with N.

Proposition 1.4. *The following two conditions are equivalent.*

(1) *The restriction of any finite-dimensional irreducible complex representation of the algebra M to N has simple multiplicities.*
(2) *The centralizer $Z(M, N)$ is commutative.*

Proof. Let V^μ and V^λ be the finite-dimensional spaces of irreducible representations of the algebras N and M, respectively. Consider the M-module $\mathrm{Hom}_N(V^\mu, V^\lambda)$. It is an irreducible $Z(M, N)$-module; thus it is one-dimensional if the centralizer is commutative.

Conversely, if there exists an irreducible representation of the centralizer $Z(M, N)$ of dimension more than one, then the multiplicity of the restriction of some representation of M to N is also greater than one. $\square$

In the next section, we will apply this criterion to the group algebras of the symmetric groups.

2. Young–Jucys–Murphy elements

From now on we consider the case

$$G(n) = S_n.$$

First let us prove that the spectrum of the restriction of an irreducible representation of S_n to S_{n-1} is simple (i.e., there are no multiplicities). The proof reproduces the idea of the classical I. M. Gelfand's criterion saying when a pair of groups — a Lie group and its subgroup — is what was later called a Gelfand pair (this means that the subalgebra consisting of those elements of the group algebra that are biinvariant with respect to the subgroup is commutative). We present this beautiful proof (I was reminded of it by E. Vinberg), because it is of very general character and uses the specific features of the symmetric group as little as possible.

Recall (see Proposition 1.4) that the spectrum of the restriction of a representation of a group to a subgroup is simple if and only if the *centralizer* of the group algebra of the subgroup in the group algebra of the whole group *is commutative.*

Theorem 2.1. *The centralizer $Z(n-1, 1) \equiv Z(\mathbb{C}[S_n], \mathbb{C}[S_{n-1}])$ of the subalgebra $\mathbb{C}[S_{n-1}]$ in $\mathbb{C}[S_n]$ is commutative.*

We begin with the following assertion.

Lemma 2.2. *Every element g of the symmetric group S_n is conjugate to the inverse element g^{-1}, i.e., there is $h \in S_n$ such that $g^{-1} = hgh^{-1}$; moreover, the element h can be chosen in the subgroup S_{n-1}.*

Proof. Indeed, it is obvious that for every k (in particular, for $k = n-1$), every permutation from S_{n-1} is conjugate to its inverse. Now let $g \in S_n$; take a permutation $h \in S_{n-1} \subset S_n$ that conjugates in S_{n-1} the permutation $g' \in S_{n-1}$ induced on $1, \dots, n-1$ by g (i.e., $g' = p_n g$, where p_n is the virtual projection; see the definition in Sec. 7) and its inverse ${g'}^{-1}$; that is, we take $h \in S_{n-1}$ such that ${g'}^{-1} = hg'h^{-1}$. Then h, regarded as an element of S_n with fixed point n, realizes the desired conjugation: $g^{-1} = hgh^{-1}$. Moreover, we can choose h to be an element of second order with fixed point n. $\square$

Recall a simple but important fact from the theory of involutive algebras.

Lemma 2.3. *A real algebra with involution $*$ is commutative if and only if all its elements are self-conjugate, i.e, if the involution is the identity automorphism.*

Proof. Let B be a $*$-algebra over $\mathbb{R}$, i.e., an algebra with a linear anti-automorphism of second order all elements of which are self-conjugate. Then any two elements commute: $ab = a^*b^* = (ba)^* = ba$, and the algebra is commutative. The converse is just as obvious: if the algebra is commutative, then the product of self-conjugate elements is also self-conjugate: $ab = ba = b^*a^* = (ab)^*$, and since these elements generate the whole algebra, all elements of the algebra are self-conjugate. $\square$

For involutive algebras over $\mathbb{C}$ the statement is slightly different: an algebra over $\mathbb{C}$ is commutative if and only if the involution is the complex conjugation with respect to every realization of the algebra as the complexification of a real algebra.

Let us continue the proof of Theorem 2.1.

Proof of Theorem 2.1. As we have seen, it suffices to check that every real element of the centralizer $Z(n-1,1) \subset \mathbb{C}[S_n]$ is self-conjugate.

Let

$$f = \sum_i c_i g_i, \qquad c_i \in \mathbb{R},$$

be an arbitrary real element of $Z(n-1,1)$; the above expansion is unique, because $\{g,\ g \in S_n\}$ is a basis in $\mathbb{C}[S_n]$. Since f commutes with every h from S_{n-1}, it follows from the uniqueness of the expansion $f = \sum_i c_i g_i$ that is does not change if we apply an inner automorphism $f \to hfh^{-1}$; as we have proved above, we can choose $h = h_i$ with $h_i g_i h_i^{-1} = g_i^{-1}$; then the summand $c_i g_i$ turns into $c_i g_i^{-1}$. Thus, along with every summand $c_i g_i$, the decomposition also contains the summand $c_i g_i^{-1}$, which means that f is a fixed point of the anti-automorphism, or that $f^* = f$. $\square$

The analysis of the whole proof leads to the following statement.

Theorem 2.4. *Let A be a finite-dimensional $*$-algebra over $\mathbb{R}$, and let B be its $*$-subalgebra; assume that in A there is a linear basis $G = \{g_i\}$ closed under the involution (i.e., $G^* = G$), and for every i there exists an orthogonal ($b^* = b^{-1}$) element $b_i \in B$ such that $b_i g_i b_i{}^* = g_i^*$. Then the centralizer of the subalgebra B in the algebra A is commutative, and thus the spectrum of the restriction of irreducible representations of the algebra A to the subalgebra B is simple.*

If A and B are the group algebras of a finite group G and its subgroup $H \subset G$, respectively, and the basis consists of elements of G, then this condition reads as follows: for every $g \in G$, there exist elements $h \in H$ and $g' \in G$ such that $h^{-1}g'h = g^{-1}$; if we can take $g' = g$, then we obtain the above condition.

This criterion in the above form can be applied in many situations. We emphasize that the above proof of the simplicity of spectrum for the symmetric groups does not use in any way the analysis of representations of S_n; and the fact itself is the first step towards the spectral analysis of the symmetric groups and is based only on elementary algebraic properties of the group. Later we will see that the simplicity of spectrum also easily follows from another fact concerning centralizers.

We will need not only the fact that the centralizer $Z(n-1,1)$ is commutative, but also a more detailed description of this centralizer as well as its relation to the Gelfand–Tsetlin algebra. We will describe the centralizer and the structure of the Gelfand–Tsetlin algebra with the help of a special basis.

For $i = 1, 2, \dots, n$, consider the following elements $X_i \in \mathbb{C}[S_n]$:

$$X_i = (1\,i) + (2\,i) + \cdots + (i-1\ i)$$

(in particular, $X_1 = 0$). We will call them the *Young–Jucys–Murphy elements* (or *YJM-elements*).

It is clear that

$$X_i = \text{sum of all transpositions in } S_i \ - \text{ sum of all transpositions in } S_{i-1}, \tag{2.1}$$

that is, X_i is the difference of an element of $Z(i)$ and an element of $Z(i-1)$. Therefore $X_i \in GZ(n)$ for all $i \le n$. In particular, the Young–Jucys–Murphy elements commute.

Let $A, B, \dots, C$ be elements or subalgebras of some algebra M; by $\langle A, B, \dots, C\rangle$ denote the subalgebra of M generated by $A, B, \dots, C$.

Theorem 2.5. *In the algebra $\mathbb{C}[S_n]$, consider its center $Z(n)$ and the center $Z(n-1)$ of the subalgebra $\mathbb{C}[S_{n-1}] \hookrightarrow \mathbb{C}[S_n]$. Then*

$$Z(n) \subset \langle Z(n-1), X_n \rangle.$$

Proof. Recall that

$$X_n = \sum_{i=i}^{n-1} (i,n) = \sum_{i \neq j;\, i,j=1}^{n} (i,j) - \sum_{i \neq j;\, i,j=1}^{n-1} (i,j).$$

The second summand lies in $Z(n-1)$, hence the first one lies in $\langle Z(n-1), X_n \rangle$. We have

$$X_n^2 = \sum_{i,j=1}^{n} (i,n)(j,n) = \sum_{i \neq j;\, i,j=1}^{n-1} (i,j,n) + (n-1)\mathbb{I}.$$

Therefore the element $\sum_{i \neq j; i,j=1}^{n-1} (i,j,n)$ lies in $\langle Z(n-1), X_n \rangle$. Adding the element

$$\sum_{i \neq j \neq k;\, i,j,k=1}^{n-1} (i,j,k)$$

from $Z(n-1)$, we obtain the following element from $Z(n)$:

$$\sum_{i \neq j \neq k;\, i,j,k=1}^{n} (i,j,k).$$

Thus we have proved that the indicator of the conjugacy class of cycles of length 3 in S_n also lies in $\langle Z(n-1), X_n \rangle$.

Apply induction and consider the general case

$$\begin{aligned} X_n \cdot \sum_{i_1,\ldots,i_{k-1}=1}^{n} (i_1,\ldots,i_{k-1},n) &= \sum_{i \neq i_s,\, s=1,\ldots,n-1} (i,n)(i_1,\ldots,i_{k-1},n) \\ &\quad + \sum_{i,i_1,\ldots,i_{k-1}} (i,i_1,\ldots,i_{k-1},n). \end{aligned}$$

Taking the sum of the first summand with the class

$$\sum_{i,j,i_1,\ldots,i_{k-1}=1}^{n} (i,j)(i_1,\ldots,i_{k-1}),$$

which lies in $Z(n-1)$, we obtain the conjugacy class in S_n of the product of a cycle of length 2 with a cycle of length k, i.e., an element from $Z(n)$. Hence the second summand, the class of cycles of length $k+1$, also lies in $\langle Z(n-1), X_n \rangle$. Again taking its sum with the element

$$\sum_{i,i_1,\ldots,i_k} (i,i_1,\ldots,i_{k-1},i_k) \in Z(n-1),$$

we obtain the conjugacy class of cycles of length $k+1$ in S_n.

Thus the classes of all one-cycle[5] permutations in S_n lie in $\langle Z(n-1), X_n \rangle$. It remains to apply the classical theorem saying that the center of the group algebra $\mathbb{C}[S_n]$ is generated by multiplicative generators — the classes of one-cycle permutations. This theorem reduces to the assertion that the power sums $\sum_{i=1}^{n} x_i^r \equiv p_r$ form a multiplicative basis in the ring of symmetric functions ([23, Chap. 1]). Thus

$$Z(n) \subset \langle Z(n-1), X_n \rangle. \qquad \square$$

[5]By one-cycle permutations, we mean permutations with one nontrivial cycle.

Corollary 2.6. *The Gelfand–Tsetlin algebra is generated by the Young–Jucys–Murphy elements:*

$$GZ(n) = \langle X_1, X_2, \dots, X_n \rangle.$$

Proof. By definition,

$$GZ(n) = \langle Z(1), \dots, Z(n) \rangle.$$

Clearly, $GZ(2) = \mathbb{C}[S_2] = \langle X_1 = 0, X_2 \rangle = \mathbb{C}$.

Assume that we have proved that

$$GZ(n-1) = \langle X_1, \dots, X_{n-1} \rangle.$$

Then we must prove that

$$GZ(n) = \langle GZ(n-1), X_n \rangle.$$

The inclusion

$$GZ(n) \supset \langle GZ(n-1), X_n \rangle$$

is obvious, hence it suffices to check that

$$Z(n) \subset \langle GZ(n-1), X_n \rangle.$$

But Theorem 2.5 implies

$$Z(n) \subset \langle Z(n-1), X_n \rangle \subset \langle GZ(n-1), X_n \rangle.$$

□

Remark 2.7. Note that the YJM-elements do not lie in the corresponding centers: $X_k \notin Z(k)$, $k = 1, \dots, n$. It might seem natural to search for a basis of $GZ(n)$ consisting of elements of the centers $Z(1), \dots, Z(n)$. However, it is a "noncentral" basis that turns out to be useful.

Theorem 2.8. *The centralizer $Z(n-1,1) \equiv Z(\mathbb{C}[S_n], \mathbb{C}[S_{n-1}])$ of the subalgebra $\mathbb{C}[S_{n-1}]$ in $\mathbb{C}[S_n]$ is generated by the center $Z(n-1)$ of $\mathbb{C}[S_{n-1}]$ and the element X_n:*

$$Z(n-1,1) = \langle Z(n-1), X_n \rangle.$$

Proof. A linear basis in the centralizer $Z(n-1,1)$ is the union of a linear basis in $Z(n-1)$ and classes of the form

$$\sum (i_1^{(1)}, \dots, i_{k_1-1}^{(1)}, n)(i_1^{(2)}, \dots, i_{k_2}^{(2)}) \dots (i_1^{(3)}, \dots, i_{k_3}^{(3)}),$$

where the sum is taken over distinct indices i_s^l that run over all numbers from 1 to $n-1$. But taking the sum of such classes with the classes

$$\sum (i_1^{(1)}, \dots, i_{k_1}^{(1)})(i_1^{(2)}, \dots, i_{k_2}^{(2)}) \dots (i_1^{(3)}, \dots, i_{k_3}^{(3)})$$

(the sum is over all indices from 1 to $n-1$) from $Z(n-1)$, as in the proof of Theorem 2.5, we obtain all classes from $Z(n)$. Hence a linear basis of $Z(n-1,1)$ can be obtained as a linear combination of elements of the bases of $Z(n-1)$ and $Z(n)$, i.e.,

$$Z(n-1,1) \subset \langle Z(n-1), Z(n) \rangle.$$

And since $Z(n) \subset \langle Z(n-1), X_n \rangle$ (by Theorem 2.5), the theorem follows. □

Theorem 2.9. *The branching of the chain $\mathbb{C}[S_1] \subset \dots \subset \mathbb{C}[S_n]$ is simple, i.e., the multiplicities of the restrictions of irreducible representations of $\mathbb{C}[S_n]$ to $\mathbb{C}[S_{n-1}]$ equal* 0 *or* 1.

Proof. Since the centralizer $Z(n-1,1)$ is commutative (because $Z(n-1,1) \subset \langle Z(n-1), X_n \rangle$), it suffices to apply the simplicity criterion from Proposition 1.4. □

Corollary 2.10. *The algebra $GZ(n)$ is a maximal commutative subalgebra of $\mathbb{C}[S_n]$. Thus in each irreducible representation of S_n, the Gelfand–Tsetlin basis is determined up to scalar factors.*

This basis is called the *Young basis.* A. Young considered it in representations, but could not describe it as a global basis, since this requires the notions of GZ-algebra and YJM-elements, which were not known then.

The Young basis is a common eigenbasis of the YJM-elements. Let v be a vector of this basis in some irreducible representation; denote by

$$\alpha(v) = (a_1, \dots, a_n) \in \mathbb{C}^n$$

the eigenvalues of $X_1, \dots, X_n$ on v. Let us call the vector $\alpha(v)$ the *weight* of v. Denote by

$$\mathrm{Spec}\,(n) = \{\alpha(v),\ v \text{ belongs to the Young basis}\}$$

the spectrum of the YJM-elements. By Theorem 2.5 and Remark 1.2, a point $\alpha(v) \in \mathrm{Spec}\,(n)$ determines v up to a scalar factor. It follows that

$$|\mathrm{Spec}\,(n)| = \sum_{\lambda \in S_n^\wedge} \dim \lambda\,.$$

In other words, the dimension of the Gelfand–Tsetlin algebra is equal to the sum of the dimensions of all pairwise nonequivalent irreducible representations.

By the definition of the Young basis, the set $\mathrm{Spec}\,(n)$ is in a natural bijection with the set of all paths (1.2) in the branching graph. Denote this correspondence by

$$T \mapsto \alpha(T), \quad \alpha \mapsto T_\alpha\,.$$

Denote by v_α the vector (unique up to a nonzero scalar factor) of the Young basis corresponding to a weight α. There is a natural equivalence relation $\sim$ on $\mathrm{Spec}\,(n)$. Write

$$\alpha \sim \beta, \quad \alpha, \beta \in \mathrm{Spec}\,(n)\,,$$

if v_α and v_β belong to the same irreducible S_n-module, or, equivalently, if the paths T_α and T_β have the same end. Clearly,

$$|\mathrm{Spec}\,(n)/\sim| = |S_n^\wedge|\,.$$

Our plan is to

(1) describe the set $\mathrm{Spec}\,(n)$,
(2) describe the equivalence relation $\sim$,
(3) calculate the matrix elements in the Young basis,
(4) calculate the characters of irreducible representations.

3. The action of generators and the algebra $H(2)$

The Coxeter generators

$$s_i = (i\ i+1), \quad i = 1, \dots, n-1,$$

of the group S_n commute except for neighbors. In [30], such generators were called *local.* Here "locality" is understood as in physics; it means that remote generators commute and hence do not affect each other. The locality manifests itself in the following property of the Young basis.

Proposition 3.1. *For any vector*

$$v_T, \quad T = \lambda_0 \nearrow \dots \nearrow \lambda_n, \quad \lambda_i \in S_i^\wedge,$$

and any $k = 1, \dots, n-1$, the vector

$$s_k \cdot v_T$$

is a linear combination of the vectors

$$v_{T'}, \quad T' = \lambda'_0 \nearrow \dots \nearrow \lambda'_n, \quad \lambda'_i \in S_i^\wedge,$$

such that

$$\lambda_i' = \lambda_i, \quad i \neq k\,.$$

In other words, the action of s_k affects only the kth level of the branching graph.

Proof. Let $i > k$. Since $s_k \in S_i$ and the module

$$\mathbb{C}[S_i] \cdot v_T$$

is irreducible, we have

$$\mathbb{C}[S_i]\, s_k \cdot v_T = \mathbb{C}[S_i] \cdot v_T = V^{\lambda_i}\,, \tag{3.1}$$

where V^{λ_i} is the irreducible S_i-module indexed by $\lambda_i \in S_i^\wedge$.

Since s_k commutes with S_i, (3.1) also holds for all $i < k$. Now it follows from (1.3) that $s_k \cdot v_T$ is a linear combination of the desired vectors. □

In the same way it is easy to show that the coefficients of this linear combination depend only on $\lambda_{k-1}, \lambda_k, \lambda_k'$, λ_{k+1} and the choice of the scalar factors in vectors of the Young basis. That is, the action of s_k affects only the kth level and depends only on levels $k-1, k$, and $k+1$ of the branching graph. More precise formulas are given in Sec. 4.

We can also easily deduce the above proposition from the obvious relations

$$s_i X_j = X_j s_i, \quad j \neq i, i+1\,. \tag{3.2}$$

The elements s_i, X_i, and X_{i+1} satisfy a more interesting (and well-known) relation

$$s_i X_i + 1 = X_{i+1} s_i\,, \tag{3.3}$$

which can obviously be rewritten as

$$s_i X_i s_i + s_i = X_{i+1}\,.$$

The action of the YJM-elements on the Young basis is also local. It readily follows from (2.1) that if

$$T = \lambda_0 \nearrow \ldots \nearrow \lambda_n$$

and

$$\alpha(T) = (a_1, \ldots, a_n),$$

then a_k is the difference of a function of λ_k and a function of λ_{k-1} for all k.

Denote by $H(2)$ the algebra generated by the elements Y_1, Y_2, and s subject to the following relations:

$$s^2 = 1, \quad Y_1 Y_2 = Y_2 Y_1, \quad sY_1 + 1 = Y_2 s\,.$$

The generator Y_2 can be excluded, because $Y_2 = sY_1 s + s$, so that the algebra $H(2)$ is generated by Y_1 and s, but technically it is more convenient to include Y_2 in the list of generators.

This algebra will play the central role in what follows. It is the simplest example of the *degenerate affine Hecke algebra* (see below). It follows directly from these relations that *irreducible finite-dimensional representations of this algebra are either one-dimensional or two-dimensional.* Indeed, since Y_1 and Y_2 commute, they have a common eigenbasis; taking any vector v of this eigenbasis and applying the involution s to v, we obtain an $H(2)$-invariant subspace of dimension at most 2. The importance of the algebra $H(2)$ is based on the following obvious yet useful fact.

Proposition 3.2. *The algebra $\mathbb{C}[S_n]$ is generated by the algebra $\mathbb{C}[S_{n-1}]$ and the algebra $H(2)$ with generators $Y_1 = X_{n-1}$, $Y_2 = X_n$, $s = s_n$, where X_{n-1} and X_n are the corresponding YJM-elements and $s_n = (n-1, n)$ is a Coxeter generator.*

Of course, the algebra $\mathbb{C}[S_n]$ is generated by the subalgebra $\mathbb{C}[S_{n-1}]$ and one generator s_n, but it is taking into account the superfluous generators X_{n-1} and X_n that allows us to use induction: each step from $n-1$ to n reduces to the study of representations of $H(2)$.

Another important property of the Coxeter generators and YJM-elements is that the relations between them are stable under shifts of indices. In [30], such relations were called *stationary.*

Remark 3.3. The *degenerate affine Hecke algebra* $H(n)$ is generated by commuting variables $Y_1, Y_2, \ldots, Y_n$ and Coxeter involutions $s_1, \ldots, s_{n-1}$ with relations (3.2), (3.3) (see [16, 13]). If we put $Y_1 = 0$, then the quotient of $H(n)$ modulo the corresponding ideal is canonically isomorphic to $\mathbb{C}[S_n]$.

4. Irreducible representations of $H(2)$

As already mentioned in Sec. 3, all irreducible representations of $H(2)$ are at most two-dimensional and have a vector v such that

$$Y_1 v = av, \quad Y_2 v = bv, \quad a, b \in \mathbb{C}\,.$$

If the vectors v and sv are linearly independent, then the relation

$$sY_1 + 1 = Y_2 s \tag{4.1}$$

implies that Y_1 and Y_2 act in the basis v, sv as follows:

$$Y_1 = \begin{pmatrix} a & -1 \\ 0 & b \end{pmatrix}, \quad Y_2 = \begin{pmatrix} b & 1 \\ 0 & a \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If $b \neq a \pm 1$, then this representation contains one of the two one-dimensional subrepresentations; denote it by $\pi_{a,b}$. If $b = a \pm 1$, then this representation contains the unique one-dimensional subrepresentation

$$Y_1 \mapsto a, \quad Y_1 \mapsto b, \quad s_1 \mapsto \pm 1,$$

in which v and sv are proportional; and, conversely, if v and sv are proportional, then

$$sv = \pm v,$$

and (4.1) implies

$$b = a \pm 1.$$

Note that always $a \neq b$, since otherwise the operators $\pi_{a,b}(Y_i)$ cannot be diagonalized and thus such representations cannot occur in the action on the Young basis. If $a \neq b$, then the operators $\pi_{a,b}$ can be diagonalized, for example, as follows:

$$Y_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad Y_2 = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}, \quad s = \begin{pmatrix} \frac{1}{b-a} & 1 - \frac{1}{(b-a)^2} \\ 1 & \frac{1}{a-b} \end{pmatrix}. \tag{4.2}$$

Let us formulate our results as a proposition which describes representations in terms of transformations of weights (i.e., eigenvectors).

Proposition 4.1. *Let*

$$\alpha = (a_1, \ldots, a_i, a_{i+1}, \ldots, a_n) \in \operatorname{Spec}(n)\,.$$

Then $a_i \in \mathbb{Z}$ and

(1) $a_i \neq a_{i+1}$ *for all* i;
(2) *if* $a_{i+1} = a_i \pm 1$, *then* $s_i \cdot v_\alpha = \pm v_\alpha$;
(3) *if* $a_{i+1} \neq a_i \pm 1$, *then*

$$\alpha' = s_i \cdot \alpha = (a_1, \ldots, a_{i+1}, a_i, \ldots, a_n) \in \operatorname{Spec}(n)$$

and $\alpha' \sim \alpha$ (see Sec. 2 for the definition of the equivalence relation $\sim$). Moreover,

$$v_{\alpha'} = \left(s_i - \frac{1}{a_{i+1} - a_i} \right) v_\alpha,$$

and the elements s_i, X_i, X_{i+1} act in the basis $v_\alpha, v_{\alpha'}$ by formulas (4.2) with Y_1 replaced by X_i and Y_2 replaced by X_{i+1}.

Recall that the transpositions s_i from claim (3) of Proposition 4.1 are Coxeter transpositions. In order to emphasize their role in the context of this section (as operations on weights α), we call them *admissible* transpositions. Admissible transpositions preserve the set $\operatorname{Spec}(n)$ and the set $\operatorname{Cont}(n)$ defined in the next section. The two cases of this proposition correspond to the cases of chain and square from Sec. 7.

Note that if $a_{i+1} \neq a_i \pm 1$, then in the basis

$$\Big\{ v_\alpha, c_i(s_i - d_i \mathbb{I}) v_\alpha \Big\},$$

where $c_i = (a_{i+1} - a_i)^{-1}$, $d_i = (1 - c_i^2)^{-1/2}$, the matrix of the transposition s_i is orthogonal:

$$s_i = \begin{pmatrix} 1/r & \sqrt{1 - 1/r^2} \\ \sqrt{1 - 1/r^2} & -1/r \end{pmatrix},$$

where $r = a_{i+1} - a_i$. In Young's papers, this difference was called the *axial* distance; it is the difference of the contents (see Sec. 5) of the corresponding boxes of Young tableaux.

5. Main theorems

In this section, we describe the set $\mathrm{Spec}\,(n)$ introduced in Sec. 2 and the equivalence relation $\sim$. Let us introduce the set $\mathrm{Cont}\,(n)$ of *content vectors* of length n.

Definition. *We say that* $\alpha = (a_1, \ldots, a_n)$ *is a content vector,*

$$\alpha = (a_1, \ldots, a_n) \in \mathrm{Cont}\,(n),$$

if α *satisfies the following conditions:*

(1) $a_1 = 0$;
(2) $\{a_q - 1, a_q + 1\} \cap \{a_1, \ldots, a_{q-1}\} \neq \varnothing$ *for all* $q > 1$ *(i.e., if* $a_q > 0$*, then* $a_i = a_q - 1$ *for some* $i < q$*; and if* $a_q < 0$*, then* $a_i = a_q + 1$ *for some* $i < q$*);*
(3) *if* $a_p = a_q = a$ *for some* $p < q$*, then*

$$\{a - 1, a + 1\} \subset \{a_{p+1}, \ldots, a_{q-1}\}$$

(i.e., between two occurrences of a *in a content vector there should also be occurrences of* $a - 1$ *and* $a + 1$*).*

It is clear that

$$\mathrm{Cont}\,(n) \subset \mathbb{Z}^n\,.$$

Theorem 5.1.

$$\mathrm{Spec}\,(n) \subset \mathrm{Cont}\,(n)\,. \tag{5.1}$$

We need the following lemma.

Lemma 5.2. *Let* $\alpha = (a_1, \ldots, a_n)$ *and* $a_i = a_{i+2} = a_{i+1} - 1$ *for some* i*, i.e.,* α *contains a fragment of the form* $(a, a + 1, a)$*. Then*

$$\alpha \notin \mathrm{Spec}\,(n)\,.$$

Proof. Let $\alpha \in \mathrm{Spec}\,(n)$. By claim (2) of Proposition 4.1,

$$s_i v_\alpha = v_\alpha, \quad s_{i+1} v_\alpha = -v_\alpha\,,$$

i.e., $s_i s_{i+1} s_i v_\alpha = -v_\alpha$, but $s_{i+1} s_i s_{i+1} v_\alpha = v_\alpha$, contradicting the Coxeter relations

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}\,. \qquad \square$$

Proof of Theorem 5.1. Let $\alpha = (a_1, \ldots, a_n) \in \mathrm{Spec}\,(n)$. Since $X_1 = 0$, we have $a_1 = 0$.

Let us verify conditions (2) and (3) by induction on n. The case $n = 2$ is trivial. Assume now that $\{a_n - 1, a_n + 1\} \cap \{a_1, \ldots, a_{n-1}\} = \varnothing$. Then the transposition of $n - 1$ and n is admissible and

$$(a_1, \ldots, a_{n-2}, a_n, a_{n-1}) \in \mathrm{Spec}\,(n)\,.$$

Hence $(a_1, \ldots, a_{n-2}, a_n) \in \mathrm{Spec}\,(n - 1)$ and, clearly,

$$\{a_n - 1, a_n + 1\} \cap \{a_1, \ldots, a_{n-2}\} = \varnothing,$$

contradicting the induction hypothesis. This proves the necessity of (2).

Assume that $a_p = a_n = a$ for some $p < n$, and let

$$a - 1 \notin \{a_{p+1}, \ldots, a_{n-1}\}\,.$$

We may assume that p is the largest possible, that is, the number a does not occur between a_p and a_n:

$$a \notin \{a_{p+1}, \ldots, a_{n-1}\}\,.$$

Then, by the induction hypothesis, the number $a+1$ occurs in the set $\{a_{p+1}, \dots, a_{n-1}\}$ at most once. Indeed, if it occurred at least twice, then, by the induction hypothesis, the number a would also occur. Thus we have two possibilities: either

$$(a_p, \dots, a_n) = (a, *, \dots, *, a)\,,$$

or

$$(a_p, \dots, a_n) = (a, *, \dots, *, a+1, *, \dots, *, a)\,,$$

where $*, \dots, *$ stands for a sequence of numbers different from $a-1$, a, $a+1$.

In the first case, applying $n-p-1$ admissible transpositions, we obtain

$$\alpha \sim \alpha' = (\dots, a, a, \dots)\,,$$

which contradicts claim (1) of Proposition 4.1.

In the second case, the same argument yields

$$\alpha \sim \alpha' = (\dots, a, a+1, a, \dots)\,,$$

which contradicts Lemma 5.2. $\square$

We will need another equivalence relation. Write

$$\alpha \approx \beta, \quad \alpha, \beta \in \mathbb{C}^n\,,$$

if β is an admissible permutation (the product of admissible transpositions) of the entries of α. Now we are ready for the appearance of Young diagrams and tableaux. Namely, we will see that vectors from $\mathrm{Cont}\,(n)$ are the *content vectors* of Young tableaux.

Recall some definitions. Denote by $\mathbb{Y}$ the Young graph (see Fig. 1).

By definition, the vertices of $\mathbb{Y}$ are Young diagrams, and two vertices ν and η are joined by a directed edge if and only if $\nu \subset \eta$ and η/ν is a single box. In this case we write $\nu \nearrow \eta$. Given a box $\square \in \eta$, the number

$$c(\square) = x\text{-coordinate of } \square - y\text{-coordinate of } \square$$

is called the *content* of $\square$ (see Fig. 2).

By $\mathrm{Tab}(\nu)$ denote the set of paths in $\mathbb{Y}$ from $\varnothing$ to ν; such paths are called *standard tableaux* or *Young tableaux.* A convenient way to represent a path $T \in \mathrm{Tab}(\nu)$,

$$\varnothing = \nu_0 \nearrow \dots \nearrow \nu_n = \nu,$$

is to write the numbers $1, \dots, n$ in the boxes $\nu_1/\nu_0, \dots, \nu_n/\nu_{n-1}$ of ν_n, respectively. Put

$$\mathrm{Tab}(n) = \bigcup_{|\nu|=n} \mathrm{Tab}(\nu)\,.$$

The following proposition can easily be checked.

Proposition 5.3. *Let*

$$T = \nu_0 \nearrow \dots \nearrow \nu_n \in \mathrm{Tab}(n)\,.$$

The mapping

$$T \mapsto (c(\nu_1/\nu_0), \dots, c(\nu_n/\nu_{n-1}))$$

is a bijection of the set of tableaux $\mathrm{Tab}(n)$ *and the set of content vectors* $\mathrm{Cont}\,(n)$ *defined at the beginning of this section. We have* $\alpha \approx \beta$, $\alpha, \beta \in \mathrm{Cont}\,(n)$, *if and only if the corresponding paths have the same end, that is, if and only if they are tableaux with the same diagram.*

Proof. The content vector of any standard Young tableau obviously satisfies conditions (1), (2), and (3) of the definition of a content vector, and these conditions uniquely determine the tableau as a sequence of boxes of the Young diagram. $\square$

In terms of Young tableaux, admissible transpositions are transpositions of numbers from different rows and columns.

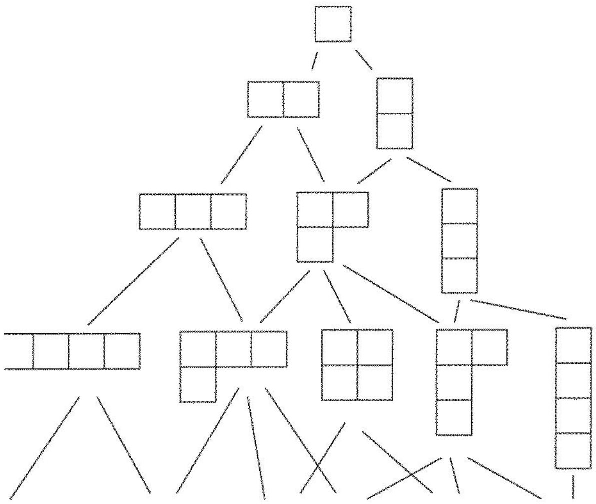


FIG. 1. The Young graph.

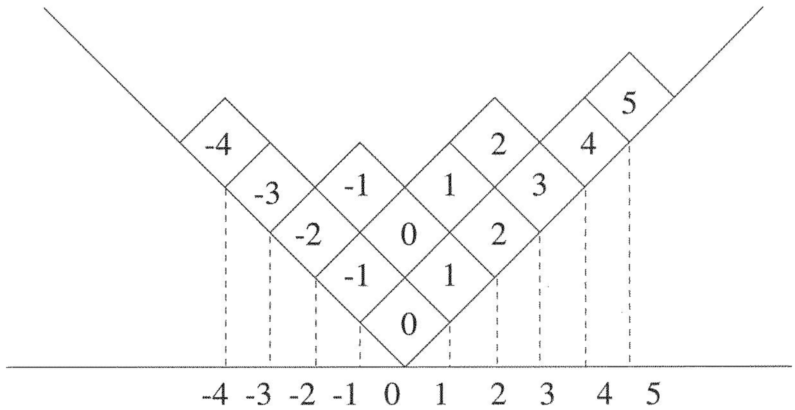


FIG. 2. Contents of boxes.

Lemma 5.4. *Any two Young tableaux $T_1, T_2 \in \mathrm{Tab}(\nu)$ with diagram ν can be obtained from each other by a sequence of admissible transpositions. In other words, if $\alpha, \beta \in \mathrm{Cont}\,(n)$ and $\alpha \approx \beta$, then β can be obtained from α by admissible transpositions.*

Proof. Let us show that by admissible transpositions we can transform any Young tableau $T \in \mathrm{Tab}(\nu)$, $\nu = (\nu_1, \dots, \nu_k)$, to the following tableau with the same diagram (and horizontal monotone numeration):

1	2	$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$	ν_1
ν_1+1	$\cdot\ \cdot\ \cdot\ \cdot\ \cdot$	$\nu_1+\nu_2$	
$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$			
$n-\nu_k+1$	$\cdots$	n	

corresponding to the content vector

$$\alpha(T^\nu) = (0, 1, 2, \dots, \nu_1 - 1, -1, 0, \dots, \nu_2 - 2, -2, -1, \dots)$$

from $\mathrm{Cont}\,(n)$. To this end, consider the last box of the last row of ν. Let i be the number written in this box of T. Transpose i and $i+1$, then $i+1$ and $i+2$, ..., and, finally, $n-1$ and n. Clearly, all these transpositions are admissible, and we obtain a tableau with the number n written in the last box of the last row. Now repeat the same procedure for $n-1, n-2, \dots, 2$. □

Corollary 5.5. *If* $\alpha \in \operatorname{Spec}(n)$ *and* $\alpha \approx \beta$, $\beta \in \operatorname{Cont}(n)$, *then* $\beta \in \operatorname{Spec}(n)$ *and* $\alpha \sim \beta$.

Remark 5.6. Our chain of transpositions from the proof of Lemma 5.4, which connects T and T^ν, is minimal possible in the following sense. Denote by s the permutation that maps T to T^ν, i.e., that associates with the number written in a given box of T the number written in the same box of T^ν. Let $\ell(s)$ be the number of inversions in s, that is,

$$\ell(s) = \#\{(i,j) \in \{1,\dots,n\} \mid i < j,\, s(i) > s(j)\}\,.$$

It is well known that s can be written as the product of $\ell(s)$ transpositions s_i and cannot be written as a shorter product[6] . It is easy to see that our chain contains precisely $\ell(s)$ admissible transpositions. In other words, $\operatorname{Cont}(n)$ is a "totally geodesic" subset of $\mathbb{Z}^n$ for the action of S_n. That is, along with any two vectors $\operatorname{Cont}(n)$ contains chains of vectors that realize the minimal path between them in the sense of the word metric with respect to the Coxeter generators.

In the proof of Lemma 5.4 we used the fact that we can transform every tableau with a given diagram into any other tableau with the same diagram using only Coxeter transpositions; it is this fact that guaranteed that vectors of the Young basis with the same diagram lie in the same representation. Thus with each irreducible representation we can associate the structure of a graph, whose vertices are vectors of the Young basis and edges are labelled by Coxeter generators and connect pairs of vectors that can be transformed into each other by the corresponding generator. These graphs generalize the Bruhat graph (order) on the group S_n.

Remark 5.7. The first author (see [2]) introduced the so-called adic transformations on the spaces of paths of graded graphs; in particular, the *Young transformation (automorphism)* on the space of infinite tableaux (i.e., paths in the Young graph). This transformation sends a tableau to the next tableau in the lexicographic order on the set of tableaux with a given diagram. Hence any two finite tableaux with the same diagram lie on the same orbit of the Young automorphism. The interval of the orbit that passes through tableaux with a given diagram starts from the tableau shown in the above figure (with horizontal monotone numeration) and ends by the tableau with vertical monotone numeration. But, of course, these orbits are not geodesic, unlike the above-defined chain of transformations, which constitutes only a part of an orbit.

Recall that the Young graph $\mathbb{Y}$ is an infinite $\mathbb{Z}$-graded graph of Young diagrams with obvious grading and set of edges. The graph consisting of the first n levels is denoted by $\mathbb{Y}_n$.

We proceed to the proof of the central theorem of the paper.

Theorem 5.8. *The Young graph* $\mathbb{Y}$ *is the branching graph of the symmetric groups; the spectrum of the Gelfand–Tsetlin algebra* $GZ(n)$ *is the space of paths in the finite graph* $\mathbb{Y}_n$, *i.e., the space of Young tableaux with* n *boxes; we have* $\operatorname{Spec}(n) = \operatorname{Cont}(n)$, *where* $\operatorname{Spec}(n)$ *is the spectrum of* $GZ(n)$ *with respect to the YJM-generators* $X_1,\dots,X_n$ *and* $\operatorname{Cont}(n)$ *is the set of content vectors; the corresponding equivalence relations coincide:* $\sim\,=\,\approx$.

Proof. As we have seen, the set of classes $\operatorname{Cont}(n)/\approx$ is the set of classes of tableaux with the same diagram. Hence

$$\#\{\operatorname{Cont}(n)/\approx\,\} = p(n),$$

where $p(n)$ is the number of partitions of the number n, i.e., the number of diagrams with n boxes. By Corollary 5.5, each equivalence class in $\operatorname{Cont}(n)/\approx$ either does not contain elements of the set $\operatorname{Spec}(n)$, or is a subset of some class in $\operatorname{Spec}(n)/\sim$. But

$$\#\{\operatorname{Spec}(n)/\sim\,\} = \#\{S_n^\wedge\} = p(n),$$

because the number of irreducible representations is equal to the number of conjugacy classes, which is again the number of partitions of n (as the number of cycle types of permutations). Therefore each class of $\operatorname{Cont}(n)/\approx$ coincides with one of the classes of $\operatorname{Spec}(n)/\sim$. In other words,

$$\operatorname{Spec}(n) = \operatorname{Cont}(n) \quad \text{and} \quad \sim\,=\,\approx\,.$$

Obviously, it follows that the graph $\mathbb{Y}$ is the branching graph of the symmetric groups. □

Thus the main theorem is proved. But the above analysis gives much more than the proof of the branching theorem; in subsequent sections we will use it to obtain an explicit model of representations (Young's orthogonal form) and sketch the derivation of the formula for characters.

[6]Simply because $\ell(s_i g) = \ell(g) \pm 1$ for all i and $g \in S_n$.

6. Young formulas

Up to now we have been considering vectors v_T of the Young basis up to scalar factors. In this section, we will specify the choice of these factors.

Let us start with the tableau T^λ defined in the proof of Lemma 5.4 (see the figure). Choose any nonzero vector v_{T^λ} corresponding to this tableau.

Now consider a tableau $T \in \mathrm{Tab}(\lambda)$ and put

$$\ell(T) = \ell(s)\,,$$

where s is the permutation that maps T^λ to T. Recall that P_T denotes the orthogonal projection onto V_T (see Sec. 1). Put

$$v_T = P_T \cdot s \cdot v_{T^\lambda}. \tag{6.1}$$

By Lemma 5.4, the permutation s can be represented as the product of $\ell(T)$ admissible transpositions. Therefore, by definition (6.1) and formulas (4.2),

$$s \cdot v_{T^\lambda} = v_T + \sum_{R \in \mathrm{Tab}(\lambda),\, \ell(R) < \ell(T)} \gamma_R\, v_R, \tag{6.2}$$

where γ_R are some rational numbers. In particular, assume that $T' = s_i T$ and

$$\ell(T') > \ell(T)\,.$$

Let

$$\alpha(T) = (a_1, \dots, a_n) \in \mathrm{Cont}\,(n)$$

be the sequence of contents of boxes in T. Then (4.2), (6.1), and (6.2) imply

$$s_i \cdot v_T = v_{T'} + \frac{1}{a_{i+1} - a_i} v_T\,. \tag{6.3}$$

And, again by (5.2),

$$s_i \cdot v_{T'} = \left(1 - \frac{1}{(a_{i+1} - a_i)^2}\right) v_T - \frac{1}{a_{i+1} - a_i} v_{T'}\,. \tag{6.4}$$

This proves the following proposition.

Proposition 6.1. *There exists a basis $\{v_T\}$ of V^λ in which the Coxeter generators s_i act according to formulas* (6.3), (6.4). *All irreducible representations of S_n are defined over the field $\mathbb{Q}$.*

Another way to prove this proposition is to verify directly that these formulas define a representation of S_n (that is, to verify the Coxeter relations).

The basis used above yields Young's *seminormal form* of V^λ. If we normalize all vectors v_T, we obtain Young's *orthogonal form* of V^λ. This form is defined over $\mathbb{R}$. Denote the normalized vectors by the same symbols v_T. Then s_i acts in the two-dimensional space spanned by v_T and $v_{T'}$ by an orthogonal matrix. Thus

$$s_i = \begin{pmatrix} r^{-1} & \sqrt{1 - r^{-2}} \\ \sqrt{1 - r^{-2}} & -r^{-1} \end{pmatrix}, \tag{6.5}$$

where

$$r = a_{i+1} - a_i\,.$$

This number is usually called the *axial distance* (see [18] and also [30]). If we write the action of the Coxeter generators s_i in the basis of standard tableaux, it looks as follows:

- if i and $i+1$ are in the same row, then s_i leaves the tableau T unchanged;
- if i and $i+1$ are in the same column, then s_i multiplies T by -1;
- if i and $i+1$ are in distinct rows and columns, then in the two-dimensional space spanned by this tableau and the tableau (which is also standard) in which the elements i and $i+1$ are swapped, s_i acts according to (6.5).

Proposition 6.2. *There exists an orthogonal basis $\{v_T\}$ of V^λ in which the generators s_i act according to formulas* (6.5).

Remark 6.3. Since the weight $\alpha(T^\lambda)$ of the vector v_{T^λ} is the maximal weight in V^λ with respect to the lexicographic order, we may call $\alpha(T^\lambda)$ the *highest weight* of V^λ and call the vector v_{T^λ} the *highest vector* of V^λ.

7. Comments and corollaries

The previous sections contain the construction of the first part of the representation theory of the symmetric groups: the description of irreducible representations, branching of representations, expressions for the Coxeter generators in representations. In particular, we have revealed the intrinsic connection between the combinatorics of Young diagrams and tableaux and the Young graph on the one hand and the representation theory of the symmetric groups on the other hand.

The further plan, which includes studying the relation to symmetric functions (characteristic map), formulas for characters, the theory of induced representations, the Littlewood–Richardson rule, the relation to the representation theory of $GL(n)$ and Hecke algebras and to asymptotic theory, can also be realized with the help of the same ideas, the main of which is an inductive approach to the series of symmetric groups.

From all these topics, in the next section we will only sketch the proof of the Murnaghan–Nakayama rule, leaving the rest to another occasion.

In this section, we will give several simple corollaries from the results obtained in Secs. 1–6. First of all, we will deduce corollaries from the branching theorem, which claims that the branching of irreducible representations of the groups S_n is described by the Young graph.

Corollary 7.1. *The multiplicity of an irreducible representation π_μ of S_n in a representation π_λ of S_{n+k} is equal to the number of paths between the diagrams λ and μ ($\lambda \vdash n+k$, $\mu \vdash n$); in particular, if $\mu \not\subset \lambda$, it is equal to* 0, *and in the general case it does not exceed $k!$, this estimate being sharp.*

Proof. Only the last claim needs to be proved. The number of tableaux in the skew diagram λ/μ does not exceed the number of different ways to add k new boxes successively to the diagram μ. If these k boxes can be added to different rows and columns, the number of these ways equals $k!$. $\square$

In particular, if $k=2$, then we have only three different cases:

(1) the multiplicity of μ in λ is equal to 0, and the vertices μ and λ are not connected in the branching graph;
(2) the multiplicity is equal to 1, and the interval connecting μ and λ is the chain

$$\mu\text{---}\nu\text{---}\lambda;$$

(3) the multiplicity is equal to 2, and the interval between μ and λ is the square

$$\begin{array}{ccccc} & \nearrow & \nu & \searrow & \\ \mu & & & & \lambda. \\ & \searrow & \eta & \nearrow & \end{array}$$

In the case of the chain, the transposition s_{l+1} multiplies all vectors of the form

$$v_T, \quad T = \ldots \nearrow \mu \nearrow \nu \nearrow \lambda \nearrow \ldots,$$

by a scalar (which is equal to ± 1 in view of the relation $s_{l+1}^2 = 1$). The action of the permutation s_{l+1} in the case of the square was considered in the previous section.

Note that the Young graph is the so-called Hasse diagram of the distributive lattice of finite ideals of the lattice $\mathbb{Z}_+ + \mathbb{Z}_+$, hence intervals in the Young graph have a standard description, and the generic interval is a Boolean algebra. *A priori* this important fact is quite nonobvious, but eventually it proved to be a corollary of the Coxeter relations. Taking it as an assumption, one can also derive the branching theory (see [30, 31]).

The next important conclusion is an abstract description of the Young–Jucys–Murphy generators based on the previous results.

Define a mapping

$$\tilde{p}_n : S_n \to S_{n-1}$$

by the following operation of deleting the last symbol:

$$\tilde{p}_n((\ldots, n, \ldots)(\ldots)\ldots(\ldots)) = ((\ldots, \cancel{n}, \ldots)(\ldots)\ldots(\ldots)),$$

where the parentheses contain the cycle decomposition of a permutation $g \in S_n$ and $\tilde{p}_n$ leaves all cycles except the first one, which contains n, unchanged and delete n from the first cycle. The mapping $\tilde{p}_n$ enjoys the following obvious properties:

(1) $\tilde{p}_n(\mathbb{I}_n) = \mathbb{I}_{n-1}$, where $\mathbb{I}_k$ is the identity in S_k;
(2) $\tilde{p}_n|_{S_{n-1}} = \mathrm{id}_{S_{n-1}} \quad (S_{n-1} \subset S_n)$;
(3) $\tilde{p}_n(g_1 h g_2) = g_1 \tilde{p}_n(h) g_2, \quad g_1, g_2 \in S_{n-1},\ h \in S_n$.

Note that conditions (1) and (2) follow from (3). Indeed, (3) implies

$$\tilde{p}_n(g\mathbb{I}) = g\tilde{p}_n(\mathbb{I}) = \tilde{p}_n(\mathbb{I}g) = \tilde{p}_n(\mathbb{I})g$$

for all $g \in S_{n-1}$, whence $\tilde{p}_n(\mathbb{I}) = \mathbb{I}$. But then $\tilde{p}_n(g) = g$ for $g \in S_{n-1}$.

By p_n denote the extension of the mapping $\tilde{p}_n$ by linearity to the group algebra $\mathbb{C}[S_n]$:

$$p_n : \mathbb{C}[S_n] \to \mathbb{C}[S_{n-1}].$$

Thus p_n is a projection of the algebra $\mathbb{C}[S_n]$ to the subalgebra $\mathbb{C}[S_{n-1}]$. For $n = 2, 3$, such a projection is not unique, but for $n \geq 4$, condition (3) uniquely determines an operation $\tilde{p}_n : S_n \to S_{n-1}$. It is easy to see that the existence of $\tilde{p}_n$ means the existence of an S_{n-1}-biinvariant partition of S_n into $(n-1)!$ sets of n elements. Such a property of a pair of groups (G, H) is not satisfied often; however, there exists a generalization of this construction to semisimple algebras (in particular, to group algebras) in the most general case.

Proposition 7.2.
$$p_n^{-1}(\{c\mathbb{I}\}) \cap Z(n-1, 1) = \{aX_n + b\mathbb{I}\}, \quad a, b, c \in \mathbb{C}.$$

In other words, the inverse image of scalars intersects the centralizer of S_{n-1} in $\mathbb{C}[S_n]$ by the two-dimensional subspace spanned by the identity and the Young–Jucys–Murphy element X_n. In particular, X_n is uniquely determined (up to scalar) as an element of the intersection

$$p_n^{-1}(\{c\mathbb{I}\}) \cap Z(n-1, 1)$$

that is orthogonal to constants.

Proof. If $p_n(\sum_{g \in S_{n-1}} c_g g) = c\mathbb{I}$, then the element $A = \sum_{g \in S_{n-1}} c_g g$ must be a linear combination of the form $A = \sum_{i=1}^n b_i(i, n)$. Such an element commutes with S_{n-1} if and only if

$$b_1 = \dots = b_{n-1} = a, \quad b_n = b, \quad \text{i.e.,} \quad A = aX_n + b\mathbb{I}. \qquad \square$$

The projection p_n allows us to define the inverse spectrum (projective limit) of the groups S_n regarded as S_{n-1}-bimodules:

$$\varprojlim(S_n, \tilde{p}_n) = \mathfrak{S};$$

the space $\mathfrak{S}$ is no longer a group, but on this space there is a left and right actions of the group S_∞ of finite permutations, because the projection $\tilde{p}_n$ commutes with the left and right actions of S_{n-1} for all n. In [20], this object was called the *space of virtual permutations*; it is studied in detail in [21]. There is a generalization of this construction to other inductive families of groups and algebras.

In conclusion we generalize the theorem on the centralizer $Z(l, k)$ of $\mathbb{C}[S_n]$ in $\mathbb{C}[S_{n+k}]$.

Theorem 7.3 [10]**.** *The centralizer*

$$Z(l, k) \equiv \mathbb{C}[S_{n+k}]^{\mathbb{C}[S_n]}$$

is generated by the center $Z(n)$ of $\mathbb{C}[S_n] \subset \mathbb{C}[S_{n+k}]$, the group S_n permuting the elements $n+1, \dots, n+k$, and the YJM-elements $X_{n+1}, \dots, X_{n+k}$.

The main case $k = 1$ is proved in Sec. 2. The general case can be proved by the same method.

Note that this method of proof is different from and simpler than that suggested in [4] and [10, 11]; namely, it turns out to be useful to consider first the subalgebra $\langle Z(n), X_{n+1}, \dots, X_{n+k} \rangle$, as in Sec. 2.

Remark 7.4. The formulas that describe the action of the symmetric group in representations associated with skew diagrams (i.e., with diagrams equal to the difference of two true Young diagrams one of which contains the other) are similar to the formulas from Sec. 6.

Indeed, let λ be a partition of $l+k$ and μ be a partition of l with $\mu \subset \lambda$. By $V^{\lambda/\mu}$ denote the $Z(l,k)$-module

$$V^{\lambda/\mu} = \mathrm{Hom}_{S_l}(V^{\mu}, V^{\lambda}).$$

It is clear that this module has an orthonormalized Young basis indexed by all Young tableaux with the skew diagram λ/μ (which is similar to the basis of the representation associated with an ordinary Young diagram). In this basis, the generators

$$X_{l+i}, \quad i = i, \dots, k,$$

of the algebra $Z(l,k)$ act by multiplication by the content of the ith box, and the Coxeter generators of the subgroup $S_k \subset Z(l,k)$ act according to formulas (6.5).

We use Theorem 7.3 in the proof of the formula for characters in the next section.

8. Characters of the symmetric groups

In this section, we give a sketch of the proof of the Murnaghan–Nakayama rule for characters of the symmetric groups. In contrast to the previous sections, we do not recall definitions of some well-known notions. The key role in the proof is played by Proposition 8.3 based on Theorem 7.3.

Recall that a Young diagram γ is called a *hook* if $\gamma = (a+1, 1^b)$ for some $a, b \in \mathbb{Z}_+$. The number b is called the *height* of the hook γ. Recall also that a skew diagram λ/μ is called a *skew hook* if it is connected and does not have two boxes on the same diagonal. In other words, λ/μ is a skew hook if the contents of all boxes of λ/μ form an interval (of cardinality $|\lambda/\mu|$) in $\mathbb{Z}$. The number of rows occupied by λ/μ minus 1 is called the *height* of λ/μ and is denoted by $\langle\lambda/\mu\rangle$. Put $k = |\lambda/\mu|$. Let $V^{\lambda/\mu}$ be the representation of S_k indexed by a skew diagram λ/μ, and let $\chi^{\lambda/\mu}$ be the corresponding character. Our aim is to prove the following well-known theorem.

Theorem 8.1. *There is the following formula:*

$$\chi^{\lambda/\mu}((12\dots k)) = \begin{cases} (-1)^{\langle\lambda/\mu\rangle} & \text{if } \lambda/\mu \text{ is a skew hook,} \\ 0 & \text{otherwise.} \end{cases} \tag{8.1}$$

Now suppose that ρ is a partition of k. Consider the following permutation from the conjugacy class corresponding to ρ:

$$(12\dots\rho_1)(\rho_1+1\ \dots\ \rho_1+\rho_2)(\dots)\dots.$$

It is clear that repeatedly applying the theorem to the action of this permutation in the Young basis, we obtain the following classical rule.

Murnaghan–Nakayama rule. *Let ρ be a partition of k. The value $\chi^{\lambda/\mu}_{\rho}$ of the character $\chi^{\lambda/\mu}$ on a permutation of cycle type ρ equals*

$$\chi^{\lambda/\mu}_{\rho} = \sum_S (-1)^{\langle S\rangle},$$

where the sum is over all sequences S,

$$\mu = \lambda_0 \subset \lambda_1 \subset \lambda_2 \dots = \lambda,$$

such that λ_i/λ_{i-1} is a skew hook with ρ_i boxes and

$$\langle S\rangle = \sum_i \langle\lambda_i/\lambda_{i-1}\rangle.$$

It is well known and can easily be proved (see, for example, [23, Chap. 1, Ex. 3.11]) that this rule is equivalent to all other descriptions of the characters, such as the relation

$$p_\rho = \sum_\lambda \chi^\lambda_\rho\, s_\lambda$$

for symmetric functions (see [23]) or the determinantal formula (see [23, 18]). Note that the theorem we are going to prove is obviously a special case of the Murnaghan–Nakayama rule.

The same proof of the following proposition was also given in [15].

Proposition 8.2. *Formula* (8.1) *is true for* $\mu = \varnothing$.

Proof. It is easy to see (for example, by induction; see also the proof of Theorem 2.1) that

$$X_2 X_3 \dots X_k = \text{sum of all } k\text{-cycles in } S_k\,. \tag{8.2}$$

The eigenvalue of (8.2) on any vector of the Young basis in V^λ equals

$$(-1)^b b!\,(k-b-1)!$$

if λ is a hook of height b, and vanishes otherwise. Clearly, the number of one-cycle permutations in S_k equals $(k-1)!$, and

$$\dim\lambda = \binom{k-1}{b}$$

if λ is a hook of height b. Taking the trace of (8.2) in V^λ proves the proposition. □

Proposition 8.3. *For any vector* v *from the Young basis of* $V^{\lambda/\mu}$,

$$\mathbb{C}[S_k]\cdot v = V^{\lambda/\mu}.$$

Proof. The space $V^{\lambda/\mu}$ is an irreducible module over the degenerate affine Hecke algebra $H(k)$. The vector v is a common eigenvector for all X_i. Thus, by the commutation relations in $H(k)$, the space

$$\mathbb{C}[S_k]\cdot v$$

is $H(k)$-invariant and hence equals $V^{\lambda/\mu}$. □

Proposition 8.4. *If* λ/μ *is not connected, then*

$$\chi^{\lambda/\mu}((12\dots k)) = 0\,.$$

Proof. Assume that $\lambda/\mu = \nu_1 \cup \nu_2$, where ν_1 and ν_2 are two skew Young diagrams that have no common edge. Let $a = |\nu_1|$, $b = |\nu_2|$. Consider the subspace of $V^{\lambda/\mu}$ spanned by all tableaux of the form λ/μ that have the numbers $1, 2, \dots, a$ in the diagram ν_1 and the numbers $a+1, \dots, k$ in the diagram ν_2. Obviously, the numbers of such tableaux equal precisely the number of tableaux of the form ν_1 and ν_2, respectively. Consider the action of the subgroup $S_a \times S_b$ of S_k on this subspace. It follows from the Young formulas that it is isomorphic, as an $S_a \times S_b$-module, to

$$V^{\nu_1} \otimes V^{\nu_2}\,.$$

By Proposition 8.3, we have an epimorphism

$$\operatorname{Ind}_{S_a\times S_b}^{S_k} V^{\nu_1}\otimes V^{\nu_2} \longrightarrow V^{\lambda/\mu}. \tag{8.3}$$

The dimensions of both sides of (8.3) equal

$$\binom{k}{a} \dim\nu_1\, \dim\nu_2\,.$$

Hence (8.3) is an isomorphism.

In the natural basis of the induced representation, the matrix of the operator corresponding to the permutation $(12\dots k)$ (as well as any other permutation that is not conjugate to any element of $S_a \times S_b$) has only zeros on the diagonal. This proves the proposition. □

Proposition 8.5. *If λ/μ has two boxes on the same diagonal, then*

$$\chi^{\lambda/\mu}((12\dots k)) = 0\,.$$

Proof. Assume that there are two such boxes. Then there is a diagram η such that

$$\mu \subset \eta \subset \lambda$$

and η/μ is a 2×2 square

$$\eta/\mu = \boxplus\,.$$

That is, $V^{\lambda/\mu}$ contains an S_4-submodule $V^{\boxplus}$. By Proposition 8.3, we have an epimorphism

$$\mathrm{Ind}_{S_4}^{S_k} V^{\boxplus} \longrightarrow V^{\lambda/\mu}\,. \tag{8.4}$$

By the branching rule and Frobenius reciprocity, the left-hand side of (8.4) contains only irreducible S_k-modules V^δ with $\boxplus \subset \delta$. In particular, δ cannot be a hook, so that

$$\chi^\delta((12\dots k)) = 0$$

by Proposition 8.2. This proves the proposition. $\square$

In fact, we have proved that under the assumptions of Proposition 8.5,

$$\mathrm{Hom}_{S_k}(V^\gamma, V^{\lambda/\mu}) = 0$$

for all hook diagrams γ.

Proposition 8.6. *Assume that λ/μ is a skew hook. Then for any hook $\gamma = (a+1, 1^b)$,*

$$\mathrm{Hom}_{S_k}(V^\gamma, V^{\lambda/\mu}) = \begin{cases} \mathbb{C}, & b = \langle\lambda/\mu\rangle, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Since translations of a skew diagram obviously preserve the corresponding S_k-module, we may assume that λ and μ are minimal, that is,

$$\lambda_1 > \mu_1, \quad \lambda_1' > \mu_1'\,.$$

Let us show that if $b < \langle\lambda/\mu\rangle$, then

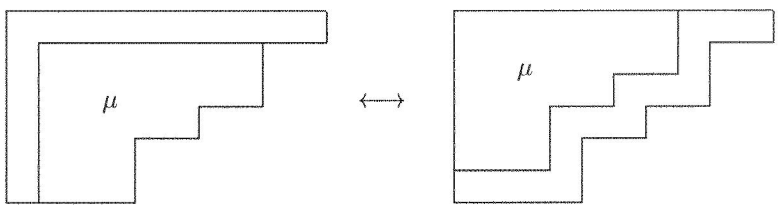
$$\mathrm{Hom}_{S_k}(V^\gamma, V^{\lambda/\mu}) = 0\,.$$

Indeed, the module V^γ contains a nonzero S_{k-b}-invariant vector, and $V^{\lambda/\mu}$ contains no such vectors, because there are no such vectors even in V^λ (this follows from the branching rule). The case $b > \langle\lambda/\mu\rangle$ is similar.

Now assume that $b = \langle\lambda/\mu\rangle$. Consider the space

$$\mathrm{Hom}_{S_k}(V^\gamma, V^\lambda)\,.$$

It is easy to see, for example, from the following picture (and the Young formulas)



that this space is the irreducible $S_{|\mu|}$-module V^μ. Therefore

$$\mathrm{Hom}_{S_k \times S_{|\mu|}}(V^\gamma \otimes V^\mu, V^\lambda) = \mathbb{C}\,,$$

whence

$$\mathrm{Hom}_{S_k}(V^\gamma, V^{\lambda/\mu}) = \mathbb{C}. \qquad \square$$

The theorem obviously follows from the propositions proved above.

Acknowledgments

We would like to thank M. Nazarov for useful information about the literature, S. Kerov and G. Olshanski for helpful discussions, and the referee for his valuable remarks.

The paper is supported by the President of Russian Federation grant for support of leading scientific schools NSh-2251.2003.1.

Translated by the authors and N. Tsilevich.

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